Bilkent University
Department of Mathematics

Problem Of The Month

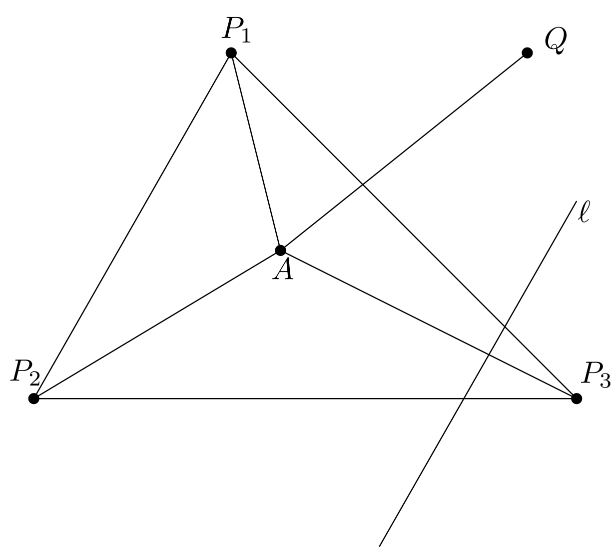
September 2016

Problem:

There are 2016 points on the plane. *Each pair* of points is connected by a segment. Find the maximal possible number of segments having no intersection with all other segments except its endpoints.

Solution: The answer is $2 \cdot 2016 - 2 = 4030$.

Let a segment that has no intersection except its endpoints be called a *special* segment. We claim that the number of special segments for $n \geq 4$ points is at most $2n - 2$. The claim is clear for $n = 4$. We will prove the case for $n > 4$ by induction.



If the n points are the vertices of a convex n-gon, there are only n special segments (namely, sides). But, clearly, $n < 2n - 2$. Otherwise, consider a point A among these n points, that is not on the boundary of the convex hull. Now, consider a triangulation of the remaining $n - 1$ points. A must lie inside some triangle of this triangulation. Let this triangle be $P_1P_2P_3$. Note that there is no point inside $P_1P_2P_3$ other than A. When we remove A, there are at most $2(n-1)-2=2n-4$ special segments, by the induction hypothesis. When we put A back, we can only add the three segments AP_1, AP_2, AP_3. Because any other segment issuing from A intersects one of the sides of the triangle $P_1P_2P_3$.

If at least one of the segments P_1P_2, P_1P_3, P_2P_3 is not special even when we remove A, then there is a segment ℓ cutting accross the triangle $P_1P_2P_3$ (One of the endpoints of this segment ℓ can coincide with a vertex of $P_1P_2P_3$, but this does not affect the subsequent analysis). This segment, then, intersects at least one of AP_1, AP_2, AP_3. Thus, when we put A back, we can only add two of the three segments AP_1, AP_2, AP_3 as special. So, the number of special segments is at most $2n-4+2=2n-2$.

If, however, all of the segments P_1P_2, P_1P_3, P_2P_3 are special when we remove A, putting A back makes at least one of these not special. Because, there is at least one point Q outside the triangle $P_1P_2P_3$ (as $n>4$) and AQ intersects at least one of P_1P_2, P_1P_3, P_2P_3. Therefore, when we put A back, even if we add all three of the segments AP_1, AP_2, AP_3 as special, we remove at least one other. So, the number of special segments is, again, at most $2n-4+3-1=2n-2$.

Now, we will show that $2n-2$ can indeed be achieved in a suitable configuration. Consider a circle and a point A outside it. Let the tangent lines from A to this circle touch it at X and Y. Take some points $A_1, A_2, \ldots, A_{n-1}$ on the small arc XY. In this configuration, the segments $A_1A_2, A_2A_3, \ldots, A_{n-1}A_1, AA_1, AA_2, \ldots, AA_{n-1}$ do not intersect any other segment. There are $2n-2$ special segments.