Bilkent University
Department of Mathematics

Problem Of The Month

May 2016

Problem:

In a class consisting of 23 students each pair of students watched a movie. A set of movies watched by a student is its *film collection*. Given that no student watched any movie more than once, what is the minimal possible number of different film collections in the class.

Solution: The answer: The minimal number of different film collections k is equal to 3.

Let us reformulate the problem in terms of graph theory. Let the edges of a complete graph on 23 vertices be properly colored (any two edges having common vertex have distinct colors). For each vertex define a collection of colors of all edges adjacent to this vertex. What is the minimal number of distinct collections?

If $k = 1$, then each vertex is adjacent to an edge colored into some particular color, say c_0. Then 23 vertices will be partitioned into pairs connected by edges colored c_0, a contradiction. If $k = 2$, suppose that the vertices $v_1, \ldots, v_l$ have the first collection and the vertices $u_1, \ldots, u_{23-l}$ have the second collection. Let the vertices v_1 and u_1 are connected by an edge colored c_0. Then each vertex is adjacent to an edge colored c_0 and again we come to the contradiction above. Now we construct an example for $k = 3$. Let us divide all vertices into three groups: $v_0, \ldots, v_{10}$, $u_0, \ldots, u_{10}$ and w. For each $0 \leq i \leq 10$ and $0 \leq j \leq 10$

the edge connecting vertices v_i and v_j we color into $c_{(i+j) mod(11)}$
the edge connecting v_i and w we color into $c_{(i+i) mod(11)}$
the edge connecting vertices u_i and u_j we color into $d_{(i+j) mod(11)}$
the edge connecting u_i and w we color into $d_{(i+i) mod(11)}$
the edge connecting v_i and u_j we color into $f_{(i+j) mod(11)}$.

Thus, by using of 33 colors $c_0, \ldots, c_{10}, d_0, \ldots, d_{10}, f_0, \ldots, f_{10}$ we have properly colored the complete graph on 23 vertices and there are only 3 different collections: each vertex v_i has the collection $\{c_0, \ldots, c_{10}, f_0, \ldots, f_{10}\}$, each vertex u_i has the collection $\{d_0, \ldots, d_{10}, f_0, \ldots, f_{10}\}$ and the vertex w has a collection $\{c_0, \ldots, c_{10}, d_0, \ldots, d_{10}\}$. Done.