Bilkent University
Department of Mathematics

Problem Of The Month

June 2015

Problem: Find the minimal value of the expression $\frac{a}{b}$ over all triples (a, b, c) of positive integers satisfying $|a^c - b!| \leq b$.

Solution: The answer: $\frac{1}{2}$.

Note that any triple $(a, b, c) = (1, 2, c)$ satisfy the given inequality. So, $\frac{a}{b}$ takes the value $\frac{1}{2}$. Now we prove that there is no other solution for $b \geq 2a$. Let $t = |a^c - n!|$. If $t > 0$, $1 = \left|\frac{a^c}{t} - \frac{b!}{t}\right|$. Since $t \leq b$, $\frac{b!}{t}$ is an integer, so $\frac{a^c}{t}$ is also an integer. Furthermore, $2a \leq b \Rightarrow a, 2a \in \{1, 2, \ldots, b\}$. At least one of a and $2a$ is different from t, so it is not canceled out from the product in $\frac{b!}{t}$. So, $a \mid \frac{b!}{t}$. Therefore, since the difference between $\frac{b!}{t}$ and $\frac{a^c}{t}$ is 1, $\gcd\left(a, \frac{a^c}{t}\right) = 1$ implying $t = a^c$. Thus, we are now left with two cases only: $|a^c - b!| = 0$ or $|a^c - b!| = a^c$. These cases reduce to

$$b! = a^c \text{ and } b! = 2a^c$$

respectively. Either way, $2a - 1 \in \{1, 2, \ldots, b\}$, so $(2a - 1) \mid b! \mid 2a^c$. Now

$$\gcd(2a - 1, 2a^c) = 1, \implies 2a - 1 = 1 \text{ and } a = 1.$$

If $a = 1$, $b! - b \leq 1 \Rightarrow b \leq 2$. So, $(a, b) = (1, 2)$ is the only solution at $b \geq 2a$. Done.