Bilkent University
Department of Mathematics

Problem Of The Month

April 2015

Problem:

In a party attended by 2015 guests among any 7 guests at most 12 handshakes had been exchanged. Determine the maximal possible total number of handshakes.

Solution: The answer is $1007 \cdot 1008 = 1015056$.

Example: Let us divide guests into two groups containing 1007 and 1008 elements and suppose that any two guests from distinct groups had exchanged handshakes and any two guests from the same group had not exchanged handshakes. Then readily conditions are satisfied and the answer is at least 1015056.

Let $f(n)$ be the maximal possible number of handshakes in a party attended by n guests. By using of the method of mathematical induction we will prove that for any $n \geq 7$ $f(n) \leq k^2$ if $n = 2k$ and $f(n) \leq k(k+1)$ if $n = 2k+1$. The case $n = 7$ is obvious. We consider two cases.

1. Suppose that the statement is proved for all $n \leq 2k-1$ and consider $n = 2k$. If the statement is not correct then $f(n) = f(2k) \geq k^2+1$ and by removing some handshakes we take $f(n) = k^2+1$. If each guest had at least $k+1$ handshakes then $f(n) \geq \frac{2k\cdot(k+1)}{2} = k^2+k > k^2+1$, a contradiction. Thus, there is a guest having at most k handshakes. If we remove him, then in total there are $2(k-1)+1$ guests and at least k^2+1-k handshakes and $k^2+1-k > (k-1)k$ which contradicts the inductive hypothesis.

2. Suppose that the statement is proved for all $n \leq 2k$ and consider $n = 2k+1$. If the statement is not correct then $f(n) = f(2k+1) \geq k^2+k+1$, and by removing some handshakes we take $f(n) = k^2+k+1$. If each guest had at least $k+1$ handshakes then $f(n) \geq \frac{(2k+1)\cdot(k+1)}{2} = k^2+k+\frac{k+1}{2} > k^2+k+1$, a contradiction. Thus, there is a guest having at most k handshakes. If we remove him, then in total there are $2k$ guests and at least k^2+1 handshakes and $k^2+1 > k^2$ which contradicts the inductive hypothesis. Done.

Finally, when $n = 2015 = 2\cdot 1007+1$ we get $f(2015) \leq 1007(1007+1) = 1015056$.