Bilkent University
Department of Mathematics

Problem Of The Month

January 2014

Problem:

All unit squares of 2014×2014 grid A are colored either white or black. Consider all 2014! colored grids obtained from A by all possible column permutations. What is the maximal possible number of distinctly colored main diagonals (diagonals starting at the lowest unit square of the leftmost column)?

Solution:

The answer: $2^{2014} - 2014$. Let C_i be the i-th column of A ($i = 1, 2, \ldots, 2014$) and the unit squares of each C_i are colored as $(c_{i,1}, c_{i,2}, \ldots, c_{i,2014})$. Let A^* be a grid obtained from A by some column permutations and $D(A^*)$ be its main diagonal. The solution is based on the following simple observation: $D(A^*)$ is constituted by unit squares of distinct columns. Below $\bar{c}_{i,j}$ denotes a color opposite to $c_{i,j}$.

First of all, we show the number of not obtainable main diagonals is at least 2014.

Case 1. All columns are distinctly colored. Then for each $l = 1, 2, \ldots, 2014$ the main diagonal colored as $(\bar{c}_{l,1}, \bar{c}_{l,2}, \ldots, \bar{c}_{l,2014})$ can not be obtained.

Case 2. The colorings of some l-th and m-th columns coincide: $C_l = (c_{l,1}, c_{l,2}, \ldots, c_{l,2014}) = (c_{m,1}, c_{m,2}, \ldots, c_{m,2014}) = C_m$. Then for each $s = 1, 2, \ldots, 2014$ the main diagonal D colored as $(\bar{c}_{l,1}, \bar{c}_{l,2}, \ldots, \bar{c}_{l,s-1}, c_{l,s}, \bar{c}_{l,s+1}, \ldots, \bar{c}_{l,2014})$, can not be obtained.

Thus, in both cases at least 2014 main diagonals are not obtainable.

Now suppose that at the beginning for each $i = 1, 2, \ldots, 2014$ all unit squares of C_i except i-th unit square are colored white and i-th unit square is colored black, in other words all unit squares of A lying on the main diagonal are black and all others are white. It can be readily shown that the total number of distinctly colored main diagonals in this case is $2^{2014} - 2014$: we can obtain all color combinations of the main diagonal except main diagonals containing 2013 black unit squares!