Bilkent University
Department of Mathematics

Problem Of The Month

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Problem:

There are N points in the space no three of which are collinear. All pairs of points are connected by $\binom{N}{2}$ line segments and each segment is colored either blue or red such that the following two conditions are held:

• there is no triangle with exactly 1 blue side

•• there are no 13 points any two connected by the same colored segment

What is the maximal possible value of N?

Solution:

The set of points A is said to be *red connected* if any points from A are connected by red segment. Let A_1 be a maximal red connected set (if there are several we choose any one of them). Consider any point c from the complement of A_1. Since A_1 is maximal, c is not connected to all points from A_1 by red segments, and since there is no triangle with exactly one blue side c is connected to all points of A_1 by blue segments. Similarly for each $k \geq 2$ we define sets A_k which are maximal red connected set from the complement of $\cup_{i=1}^{k-1} A_i$ until $|\cup_{i=1}^{k} A_i| = N$. As it was shown above, any point c from the complement of A_i is connected to all points of A_i by blue segments. Thus, any two points from A_i and A_j, $i \neq j$ are connected by blue segments. Since no 13 points are connected by the same colored segments $|A_i| \leq 12$ and $k \leq 12$. Therefore, $N \leq 12 \cdot 12 = 144$. The bound $N = 144$ is achieved in the straightforward example when $|A_i| = 12$ for each $k = 1, 2, \ldots, 12$. Done.