Bilkent University
Department of Mathematics

Problem Of The Month

March 2013

Problem:

Find all nonnegative integers a and b such that both $\dfrac{ab^2-b}{a^2+b^2}$ and $\dfrac{ba^2-a}{a^2+b^2}$ are integers.

Solution:

If $a=0$ then the only possibility is $b=1$ and if $b=0$ the only possibility is $a=1$. Now suppose that $a,b>0$ and the integer factorizations of a and b contain factors p^s and p^t respectively, and without loss of generality suppose that $s \leq t$. Then p^{2s} divides ba^2 and a^2+b^2 but does not divide a which contradicts the fact that $\frac{ba^2-a}{a^2+b^2}$ is integer. Therefore, a and b are relatively prime numbers and readily b and a^2+b^2 are also relatively prime numbers. Now since $\frac{ab^2-b}{a^2+b^2} = \frac{b(ab-1)}{a^2+b^2}$ is an integer number a^2+b^2 divides $ab-1$. Since $ab-1 < ab \leq 2ab \leq a^2+b^2$ we conclude that $ab=1$. Thus, $(a,b)=(0,1),(1,0),(1,1)$.