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Problem Of The Month

February 2013

Problem:

An airway company operates *one way* flights between some pairs of cities of a country consisting of n cities so that any city is reachable from any city (not necessarily directly). Suppose that in any such flight scheduling the shortest (including minimal number of flights) closed trip visiting each city (at least once) consists of at most 2013 flights. Find the maximal possible value of n.

Solution:

The answer: $n = 88$.

First of all, let us show that for $n \geq 89$ there is a flight scheduling for which the shortest closed trip consists more than 2013 flights. Let $A_1, \ldots A_{89}, \ldots, A_n$ be cities, and flights are scheduled as:

from A_i to A_{i+1} for $i = 1, \ldots, 43$

from A_{44} to A_i for $i = 45, \ldots, 89$

from A_i to A_1 for $i = 45, \ldots, 89$

also if $n > 89$ from A_1 to A_{90}, from A_i to A_{i+1} for $i = 90, \ldots, n-1$ and from A_n to A_1.

I can be readily shown that the closed trip visiting all cities consists of at least $45 \cdot 45 = 2025 > 2013$ flights.

Now let us show that for $n = 88$ the shortest closed trip visiting each city consists of at most 2013 flights. For any two cities A_i and A_j let $d(A_i, A_j)$ be the minimal number of flights necessary for reaching A_j from A_i and $max_{i,j} d(A_i, A_j) = d(A_l, A_m) = p$, where the maximum is taken over all possible pairs (i, j). Consider a trip from A_l to A_m. Since it is a shortest trip from A_l to A_m it visits each city at most once and therefore, visits exactly p distinct cities. From A_m we will visit a city not visited during the trip from A_l to A_m by using at most p flights. After that we will visit a new city not visited before, and so on until all cities are visited. Then the total number of flights will be at most $p + p(88 - p) = p(89 - p) \leq 44 \cdot 45 = 1980 \leq 2013$. Done.