Bilkent University
Department of Mathematics

Problem Of The Month

April 2012

Problem:

Let $x_1, x_2, \ldots, x_n$ be n points on the plane no three of which are collinear and $f(x_k, x_l, x_m)$ be the total number of points lying strictly outside of the triangle with vertices x_k, x_l, x_m. Show that

$$\sum f(x_k, x_l, x_m) \geq \frac{3n-9}{4}\binom{n}{3}$$

where the summation is taken over all non-ordered triples (x_k, x_l, x_m).

Solution:

It can be readily shown that $\sum f(x_k, x_l, x_m)$ is the total number of (triangle, point) pairs where the point lies outside of the triangle. Now we note that the contribution of any four points to the sum is at least 3 (when one point lies inside of the triangle constituted by other three points). Therefore,

$$\sum f(x_k, x_l, x_m) \geq 3 \cdot \binom{n}{4} = \frac{3n-9}{4}\binom{n}{3}.$$