Bilkent University
Department of Mathematics

Problem Of The Month

March 2011

Problem:

There are n points in the space. Some pairs of points are connected by nonintersecting segments. For each point p, let $C(p)$ be the union of p with the set of all points directly connected to p. There are two colors: white and black. At the beginning all points are colored white. At each move we chose a point p and change the color of each point in $C(p)$. Prove that in finite number of moves all points can be colored black.

Solution:

The set of all points $\{p_1, p_2, \ldots, p_n\}$ we denote by A. Proof by induction.

The case $n = 1$ is clear.

Inductive hypothesis: n points can be colored black. For any fixed point p_i, $i = 1, \ldots, n+1$ let $I(i)$ be the set of moves coloring points $A - p_i$ black. If for some i I_i also colors p_i black, we are done. Suppose not.

Case 1. $n+1$ is even. It can be readily seen that the series of moves $I(1), I(2), \ldots, I(n+1)$ colors all $n+1$ points black.

Case 2. $n+1$ is odd. Since $n+1$ is odd, there is at least one point $q \in A$ having an odd number of elements in $C(q)$. Let $C(q) = \{q, q_1, q_2, \ldots, q_{2l}\}$. Now the series of moves $q, I(q), I(q_1), I(q_2), \ldots, I(q_{2l})$ colors all $n+1$ points black.

The proof is completed.