Bilkent University
Department of Mathematics

Problem Of The Month

September 2010

Problem:

Let $\Delta(a,b,c) = max(|a-b|, |b-c|, |c-a|)$. We say that a triple (a,b,c) is good, if for all $x \in [0,1]$ we have $-1 \leq ax^2 + bx + c \leq 1$. Find a minimal constant C such that for all good triples $\Delta(a,b,c) \leq C$.

Solution:

Since $f(0) = c, f(1/2) = a/4 + b/2 + c$ and $f(1) = a + b + c$,
we get $a = 2f(0) + f(1) - 4f(1/2)$, $b = -3f(0) - f(1) + 4f(1/2)$ and $c = f(0)$.
Therefore, $|a-b| = |5f(0) + 3f(1) - 8f(1/2)| \leq 16$, $|b-c| = |-4f(0) - f(1) + 4f(1/2)| \leq 9$ and $|c-a| = |-f(0) - 2f(1) + 4f(1/2)| \leq 7$.
Thus, $\Delta(a,b,c) \leq 16$.
Straightforward calculations show that for all $x \in [0,1]$ $-1 \leq 8x^2 - 8x + 1 \leq 1$. Therefore, $(8,-8,1)$ is a good triple. Since $\Delta(8,-8,1) = 16$, $C = 16$.