Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

May 2010

Problem:

Let a, b and c be positive real numbers and $s = abc$. Find the minimal number L satisfying

$$\frac{a^3 - s}{2a^3 + s} + \frac{b^3 - s}{2b^3 + s} + \frac{c^3 - s}{2c^3 + s} \leq L$$

Solution:

We prove that $L = 0$. Let

$$f(a, b, c) = \frac{a^3 - s}{2a^3 + s} + \frac{b^3 - s}{2b^3 + s} + \frac{c^3 - s}{2c^3 + s}$$

Since $f(t, t, t) = 0$, $L \geq 0$.

Let us prove that $L \leq 0$, equivalently $f(a, b, c) \leq 0$. Since

$$f(a, b, c) = \frac{-3a^3s^2 - 3b^3s^2 - 3c^3s^2 + 9s^3}{(2a^3 + s)(2b^3 + s)(2c^3 + s)} = \frac{3s^2(3s - a^3 - b^3 - c^3)}{(2a^3 + s)(2b^3 + s)(2c^3 + s)}$$

we have to establish the inequality $3s - a^3 - b^3 - c^3 \leq 0$, which is an arithmetic-geometric inequality for a^3, b^3, c^3. Done.