Bilkent University
Department of Mathematics

Problem Of The Month

March 2010

Problem:

Find all functions $f : \mathbf{Q} \longrightarrow \mathbf{Q}$ satisfying $f(1) = 2$ and $f(xy) + f(x+y) = f(x)f(y) + 1$, where $\mathbf{Q}$ denotes the set of all rational numbers.

Solution:

Put $y = 1$ in the functional equation: $f(x) + f(x+1) = f(x)f(1) + 1$. Since $f(1) = 2$, we get $f(x+1) = f(x) + 1$ and $f(x+q) = f(x) + q$ for all integers q. Thus, $f(p+1) = p+1$ for all integers p. For integer p and natural q, put $x = \frac{p}{q}$ and $y = q$ in the functional equation: $f(\frac{p}{q} \cdot q) + f(\frac{p}{q} + q) = f(\frac{p}{q})f(q) + 1$ or $f(p) + f(\frac{p}{q}) + q = f(\frac{p}{q})(q+1) + 1$. Therefore, $f(\frac{p}{q}) = \frac{p+q}{q} = 1 + \frac{p}{q}$. The function $f(\frac{p}{q}) = 1 + \frac{p}{q}$ satisfies the conditions.