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Problem Of The Month

November 2008

Problem:

An integer sequence $\{a_1, a_2, \dots\}$ is said to be *white*, if for all $n > 2008$, a_n is equal to the total number of those indices i, $1 \le i \le n-1$ for which $a_i + i \ge n$. An integer L is an *important* element of the sequence $\{a_1, a_2, \dots\}$, if $a_j = L$ for infinitely many different indices j. What is the maximal possible number of *important* elements of a *white* sequence?

Solution:

Define $K = max\{a_1, a_2, \dots, a_{2008}, 2008\}$. Let us prove that $a_n \le K$ for all n:

1. If $1 \le n \le 2009$ then $a_n = \sum_{i:1\le i\le n-1, a_i+i\ge n} 1 \le \sum_{i:1\le i\le 2008} 1 \le 2008.$

2. Suppose that $a_n \le K$ for all $1 \le n \le k$. Then

$$a_{k+1} = \sum_{i:1\le i\le k, a_i+i\ge k+1} 1 \le \sum_{i:1\le i\le k, i\ge k+1-K} 1 \le k - (k+1-K) + 1 = K.$$

Now note that

a. the value of a_n is determined only by the preceding K terms: for $a_{n-j} + n - j \ge n$ implies that $j \le K$.

b. there are only finite number of possible blocks consisting of K consecutive terms.

Therefore, any *white* sequence becomes periodic starting from some index l, and consequently any *white* sequence has at least one *important* element. Let M be the maximal *important* element of the sequence: $M = max\{a_i, i > l\}$.
From now all indices are supposed to be greater than l. Note that if $a_i = M$ then $a_{i-M} = M$, since a_i is determined by the preceding M terms.

We prove that for all $i > l$, a_i takes at most 2 values: either M or $M - 1$. On the contrary, suppose that some terms are less than $M - 1$. Consider indices n and $k > n$ such that $a_{n+M} = M, a_n = M$ and $a_k < M - 1$ such that $k - n$ is <u>minimal</u> . Then $a_{k-1} < M$ and the inequalities $a_k < M - 1$ and $a_{k-1} < M$ imply that $a_{k+M-1} < M - 1$. We came to the contradiction with the minimality of $k - n$, since the difference between indices $k + M - 1$ and $n + M$ is less than $k - n$ but $a_{n+M} = M, a_{k+M-1} < M - 1$.

The sequence starting with $a_1 = a_2 = \cdots = a_{2006} = 0, a_{2007} = 2, a_{2008} = 1$ is a *white* sequence with two *important* elements 1 and 2.