Bilkent University
Department of Mathematics

PROBLEM OF THE MONTH

September 2008

Problem:

Let a, b, c, d be positive integer numbers satisfying

$$a = \frac{b^b - c^c}{d^d}$$

What is the possible minimal value of a?

Solution:

The answer is $a = 3$.
Since a is natural $b \geq c + 1$ and $b \geq d + 1$. Therefore, $b \geq 2$. If $b = 2$ then the only possibility is: $c = d = 1$ and $a = 3$. Suppose that $b \geq 3$ and $a \leq 2$. Then

$$b^b = a \cdot d^d + c^c \leq 2 \cdot (b-1)^{b-1} + (b-1)^{b-1} = 3(b-1)^{b-1} < 3b^{b-1}$$

yielding $b < 3$. The contradiction shows that the minimal value of a is 3.