**Bilkent University**
**Department of Mathematics**

# PROBLEM OF THE MONTH

May 2008

**Problem:**

Find the minimum of

$$\frac{1+a+b+c}{3+2a+b}-\frac{c}{b}$$

where $a, b$ and $c$ are real numbers such that all roots of the equation $x^3-ax^2+bx-c=0$ are real positive numbers.

**Solution:**

We prove that the minimum is $\frac{1}{3}$. Let $x_i > 0$ for $i = 1, 2, 3$ be the roots of the equation $x^3-ax^2+bx-c=0$. By Vieta's theorem

$$x_1+x_2+x_3=a, \quad x_1x_2+x_2x_3+x_1x_3=b, \quad x_1x_2x_3=c.$$

Then

$$A=\frac{1+a+b+c}{3+2a+b}-\frac{c}{b}$$

$$=\frac{(x_1+1)(x_2+1)(x_3+1)}{(x_1+1)(x_2+1)+(x_2+1)(x_3+1)+(x_1+1)(x_3+1)}-\frac{x_1x_2x_3}{x_1x_2+x_2x_3+x_1x_3}$$

$$=\frac{1}{\frac{1}{x_1+1}+\frac{1}{x_2+1}+\frac{1}{x_3+1}}-\frac{1}{\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3}}=\frac{\frac{1}{x_1(x_1+1)}+\frac{1}{x_2(x_2+1)}+\frac{1}{x_3(x_3+1)}}{(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3})(\frac{1}{x_1+1}+\frac{1}{x_2+1}+\frac{1}{x_3+1})}$$

Without loss of generality we can assume that $0 < x_1 \leq x_2 \leq x_3$. Then

$\frac{1}{x_1} \geq \frac{1}{x_2} \geq \frac{1}{x_3} > 0$ and $\frac{1}{x_1+1} \geq \frac{1}{x_2+1} \geq \frac{1}{x_3+1} > 0$.

By Chebyshev's inequality

$$(\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3})(\frac{1}{x_1+1}+\frac{1}{x_2+1}+\frac{1}{x_3+1}) \leq \frac{1}{3}(\frac{1}{x_1(x_1+1)}+\frac{1}{x_2(x_2+1)}+\frac{1}{x_3(x_3+1)})$$

Therefore, $A \geq 3$. The minimum 1/3 of $A$ is reached for $x_1 = x_2 = x_3 = t > 0$ or $a = 3t$, $b = 3t^2$, $c = t^3$.