Bilkent University
Department of Mathematics

Problem Of The Month

March 2008

Problem:
Let T be a set of natural numbers such that for any $a, b \in T$, $a^2 - ab + b^2$ divides a^2b^2. Prove that the set T is finite.

Solution:

Let d be the greatest common divisor of a and b: $d = (a, b)$. Then $a = da_1$ and $b = db_1$ and $(a_1, b_1) = 1$. Since $d^2(a_1^2 - a_1b_1 + b_1^2)$ divides $d^4a_1^2b_1^2$ we get $a_1^2 - a_1b_1 + b_1^2$ divides $d^2a_1^2b_1^2$. $(a_1, b_1) = 1$ implies that $(a_1^2 - a_1b_1 + b_1^2, a_1b_1) = 1$. Therefore, $a_1^2 - a_1b_1 + b_1^2$ divides d^2 or $a^2 - ab + b^2$ divides d^4. Since $d \leq a$ we get $a^2 - ab + b^2 \leq a^4$. Let us fix any $a \in T$. Then $a^2 - ab + b^2 \leq a^4$ implies that b can take only a finite number of distinct values. Done.