**Bilkent University**
**Department of Mathematics**

# Problem Of The Month

October 2007

**Problem:**
Find all non-negative rational solutions of the equation:

$$x^y \cdot y^x = y^y$$

**Solution:**

The answer: $(x, y) = (0, 0)$ or $(k^{k-1}, k^k)$, where $k$ is a natural number.

Let $(x, y)$ be a rational nonzero solution. Consider a substitution $t = \frac{y}{x}$. Then $x^{tx}(tx)^x = (tx)^{(tx)}$ or $x = t^{t-1}$. If $t - 1 = k$ then $x = (k + 1)^k$. Let us show that $k$ is a natural number. Take $x = \frac{m}{n}$ and $k = \frac{p}{q}$ where both fractions are irreducible: $(m, n) = 1$ and $(p, q) = 1$. Then our equation becomes

$$m^q q^p = n^q (p + q)^p$$

Since $(m, n) = 1$, $n^q | q^p$ and since $(p, q) = 1$, $((p + q)^p, q^p) = (p + q, q)^p = (p, q)^p = 1$. Therefore, $q^p | n^q$ and as a result, $q^p = n^q$. Suppose that $q > 1$. Then in the prime decomposition of $q^p$ each factor has a power divisible by both $p$ and $q$, so divisible by $pq$. Therefore, in the prime decomposition of $q$ each factor has a power divisible by $q$. Contradiction, since $q < 2^q$. Thus, $q = 1$ and $x = k^{k-1}$ and $y = k^k$. Straightforward check shows that $(0, 0)$ and $(k^{k-1}, k^k)$ are solutions. Done.