Bilkent University
Department of Mathematics

Problem Of The Month

October 2006

Problem:

Let x_1, x_2, x_3 and x_4 be real numbers satisfying the following equations:

$x_1 + x_2 + x_3 + x_4 = 0$

and

$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1.$

Find the maximal possible value of the expression $x_1^3 + x_2^3 + x_3^3 + x_4^3$.

Solution:

The answer is $\frac{\sqrt{3}}{3}$.

By substituting $x_4 = -(x_1 + x_2 + x_3)$ into $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$ we get $x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1 = \frac{1}{2}$. Since $x_1x_2 + x_2x_3 + x_3x_1 \leq x_1^2 + x_2^2 + x_3^2$ (Cauchy-Schwarz inequality for vectors (x_1, x_2, x_3) and (x_2, x_3, x_1)) we readily get $x_1^2 + x_2^2 + x_3^2 \geq \frac{1}{4}$. Therefore, $x_4^2 = 1 - (x_1^2 + x_2^2 + x_3^2) \leq 1 - \frac{1}{4} = \frac{3}{4}$ or $-\frac{\sqrt{3}}{2} \leq x_4 \leq \frac{\sqrt{3}}{2}$. By symmetry,

$$-\frac{\sqrt{3}}{2} \leq x_i \leq \frac{\sqrt{3}}{2} \tag{1}$$

for each $i = 1, 2, 3, 4$.

Denote $x_1^3 + x_2^3 + x_3^3 + x_4^3$ by A. Then

$$A = x_1^3 + x_2^3 + x_3^3 - (x_1 + x_2 + x_3)^3 = -3(x_1 + x_2)(x_1 + x_3)(x_2 + x_3).$$

Let us consider the signs of three expressions: $(x_1 + x_2)$, $(x_1 + x_3)$ and $(x_2 + x_3)$(if any one of them is equal to zero then $A = 0$) . If all three expressions are positive or exactly one of them is positive, then $A < 0$.

If exactly two expressions are positive, say $(x_1 + x_2) < 0$, $(x_2 + x_3) > 0$ and $(x_1+x_3) > 0$ then $A = -3(x_1+x_2)(x_1+x_3)(x_2+x_3) = 3(-x_1-x_2)(x_1+x_3)(x_2+x_3) \le 3(\frac{-x_1 - x_2 + x_1 + x_3 + x_2 + x_3}{3})^3 = \frac{8}{9}x_3^3 \le \frac{\sqrt{3}}{3}$ (by AM-GM inequality and by (1)).

If all three expressions are negative, then $A = -3(x_1+x_2)(x_1+x_3)(x_2+x_3) = 3(-x_1-x_2)(-x_1-x_3)(-x_2-x_3) \le 3(\frac{-x_1 - x_2 - x_1 - x_3 - x_2 - x_3}{3})^3 = 3(\frac{-2(x_1 + x_2 + x_3)}{3})^3$ $= \frac{8}{9}x_4^3 \le \frac{\sqrt{3}}{3}$ (by AM-GM inequality and by (1)).

Now we note that at $x_1 = \frac{\sqrt{3}}{2}, x_2 = x_3 = x_4 = -\frac{\sqrt{3}}{2}$ $A = \frac{\sqrt{3}}{3}$.