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Problem Of The Month

April 2006

Problem: Find all functions $f : \mathbf{Z} \to \mathbf{Z}$ satisfying

$$f(n) - f(n + f(m)) = m \tag{1}$$

for all integer values of m and n.

Solution:

A. Let us prove that f takes each value at most once. Indeed, since $f(n + f(m_1)) = f(n) - m_1$ and $f(n + f(m_2)) = f(n) - m_2$, $f(m_1) = f(m_2)$ implies that $m_1 = m_2$.

B. Let us prove that f takes any integer number $-m$. Indeed, if we put $m = 0$ in (1) we get $f(n + f(0)) = f(n)$. Therefore, by (A) $f(0) = 0$. And if we put $n = 0$ in (1) we get $f(f(m)) = -m$.

C. $f(n + m) = f(n + f(f(-m))) = f(n) - f(f(-m)) = f(n) + f((f(f - m))) = f(n) + f(m)$ implies that $f(n) = cn$. If we put $f(n) = cn$ into equation $f(f(m)) = -m$ from (B) we get $c^2 n = -n$ or $c^2 = -1$. Impossible: there is no function f satisfying (1).