Bilkent University
Department of Mathematics

Problem Of The Month

February 2006

Problem: Find all real numbers x and y satisfying the following equations:

$$8x^2 - 2xy^2 = 6y = 3x^2 + 3x^3y^2$$

Solution: Put $A = 8x^2 - 2xy^2$, $B = 6y$ and $C = 3x^2 + 3x^3y^2$.

$A = C$ implies that $x(5x - 2y^2 - 3x^2y^2) = 0$. Therefore, either $x = 0$ (and consequently $y = 0$) or $x \neq 0$. If $x \neq 0$, then $y^2 = \dfrac{5x}{2+3x^2}$ and $x > 0$. But $x > 0$ implies $y > 0$ since $B = C$ implies $2y = x^2 + x^3y^2$.

Square both sides of $2y = x^2 + x^3y^2$:

$$4y^2 = x^4 + x^6y^4 + 2x^5y^2$$

Replace y^2 by $\dfrac{5x}{2+3x^2}$:

$$\frac{20x}{2+3x^2} = x^4 + \frac{25x^8}{(2+3x^2)^2} + \frac{10x^6}{2+3x^2}$$

since $x \neq 0$ we get

$20(2+3x^2) = x^4(2+3x^2)^2 + 25x^7 + 10x^5(2+3x^2)$ or

$16x^7 + 8x^5 + x^3 - 15x^2 - 10 = 0$.

After factorization:

$(x-1)(16x^6 + 16x^5 + 24x^4 + 24x^3 + 25x^2 + 10x + 10) = 0$

The only positive solution is $x = 1$(in this case $y = 1$).

Thus, we have two solutions: $(0, 0)$ and $(1, 1)$.