Phys 438/538 Atomic, Molecular and Optical Physics

Second Midterm Examination

19 April 2016

Duration: 110 Minutes, Closed notes/books

1) (25 points) Wigner-Eckart theorem for electric dipole transitions
Consider the dipole pumping from the $2p$, $m = -1$ state upward to all possible $3d$ sub-levels (no spins involved). Using Wigner-Eckart theorem, determine the relative probabilities (branching ratios) for the available transitions, and the associated absorbed photon polarizations for each case (show on a level diagram). Use the Clebsch-Gordan coefficient table supplied at the back.

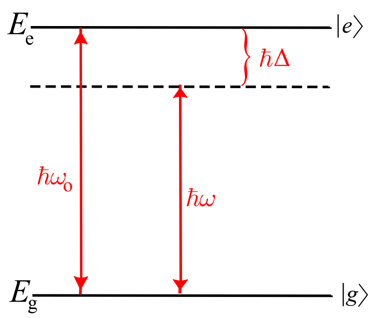
2) (30 points) f-sum Rule
For an atom governed by the Hamiltonian, $H_0 = \frac{p^2}{2\mu} + V(r)$, having eigenenergies E_i, derive the f-sum rule $\sum_f f_{fi} = 1$, where

$$f_{fi} = \frac{2\mu(E_f - E_i)}{3\hbar^2} \left|\langle f|\vec{r}|i\rangle\right|^2 .$$

Give all the details of your work.

3) (45 points) Light Shift (AC Stark Effect)

A two-level atom having an energy separation $\hbar\omega_0 = E_e - E_g$ is illuminated with an electromagnetic field of angular frequency ω, giving rise to a detuning of $\Delta \equiv \omega_0 - \omega$. This forms manifolds of the form [n photons and the atom in the excited state, $|e\rangle$] and the [$n+1$ photons and the atom in the ground state, $|g\rangle$]: the uncoupled part of the manifolds can be expressed with the Hamiltonian $H_0 = \hbar\Delta|e\rangle\langle e|$, and their interaction with $H^{'} = -\frac{\mathcal{V}}{2}\left(|e\rangle\langle g| + |g\rangle\langle e|\right)$, in terms of the resonant Rabi energy, $\mathcal{V}$. Here, we suppress the photonic states and neglect any decay terms (spontaneous emission etc).



A two-level system

(a) Determine the *exact* energy eigenvalues for the so-called *dressed atom*, and plot them as a function of Δ.

(b) Determine the corresponding eigenvectors using θ, where $\tan(\theta) \equiv \frac{\mathcal{V}}{\hbar\Delta}$.

Simplify your final expressions. If you like, you can set $\hbar \to 0$.
Some useful trigonometric identities:
$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$, $\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$, $\sin(\theta) = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$.

43. CLEBSCH-GORDAN COEFFICIENTS, SPHERICAL HARMONICS, AND d FUNCTIONS

Note: A square-root sign is to be understood over *every* coefficient, *e.g.*, for −8/15 read $-\sqrt{8/15}$.

Notation:

		J	J	...
		M	M	...
m_1	m_2	Coefficients		
m_1	m_2			
.	.			

1/2×1/2

m_1	m_2	$J=1$, $M=+1$
+1/2	+1/2	1

m_1	m_2	$J=1$, $M=0$	$J=0$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=1$, $M=-1$
−1/2	−1/2	1

1×1/2

m_1	m_2	$J=3/2$, $M=+3/2$
+1	+1/2	1

m_1	m_2	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+1	−1/2	1/3	2/3
0	+1/2	2/3	−1/3

m_1	m_2	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
0	−1/2	2/3	1/3
−1	+1/2	1/3	−2/3

m_1	m_2	$J=3/2$, $M=-3/2$
−1	−1/2	1

2×1/2

m_1	m_2	$J=5/2$, $M=+5/2$
+2	+1/2	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2	−1/2	1/5	4/5
+1	+1/2	4/5	−1/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$
+1	−1/2	2/5	3/5
0	+1/2	3/5	−2/5

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$
0	−1/2	3/5	2/5
−1	+1/2	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1	−1/2	4/5	1/5
−2	+1/2	1/5	−4/5

m_1	m_2	$J=5/2$, $M=-5/2$
−2	−1/2	1

3/2×1/2

m_1	m_2	$J=2$, $M=+2$
+3/2	+1/2	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2	−1/2	1/4	3/4
+1/2	+1/2	3/4	−1/4

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$
+1/2	−1/2	1/2	1/2
−1/2	+1/2	1/2	−1/2

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
−1/2	−1/2	3/4	1/4
−3/2	+1/2	1/4	−3/4

m_1	m_2	$J=2$, $M=-2$
−3/2	−1/2	1

$$Y_1^0=\sqrt{\frac{3}{4\pi}}\cos\theta$$

$$Y_1^1=-\sqrt{\frac{3}{8\pi}}\sin\theta\, e^{i\phi}$$

$$Y_2^0=\sqrt{\frac{5}{4\pi}}\Big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\Big)$$

$$Y_2^1=-\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{i\phi}$$

$$Y_2^2=\frac{1}{4}\sqrt{\frac{15}{2\pi}}\sin^2\theta\, e^{2i\phi}$$

2×1

m_1	m_2	$J=3$, $M=+3$
+2	+1	1

m_1	m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+2	0	1/3	2/3
+1	+1	2/3	−1/3

m_1	m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2	−1	1/15	1/3	3/5
+1	0	8/15	1/6	−3/10
0	+1	2/5	−1/2	1/10

m_1	m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$
+1	−1	1/5	1/2	3/10
0	0	3/5	0	−2/5
−1	+1	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	2/5	1/2	1/10
−1	0	8/15	−1/6	−3/10
−2	+1	1/15	−1/3	3/5

m_1	m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1	−1	2/3	1/3
−2	0	1/3	−2/3

m_1	m_2	$J=3$, $M=-3$
−2	−1	1

1×1

m_1	m_2	$J=2$, $M=+2$
+1	+1	1

m_1	m_2	$J=2$, $M=+1$	$J=1$, $M=+1$
+1	0	1/2	1/2
0	+1	1/2	−1/2

m_1	m_2	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+1	−1	1/6	1/2	1/3
0	0	2/3	0	−1/3
−1	+1	1/6	−1/2	1/3

m_1	m_2	$J=2$, $M=-1$	$J=1$, $M=-1$
0	−1	1/2	1/2
−1	0	1/2	−1/2

m_1	m_2	$J=2$, $M=-2$
−1	−1	1

3/2×1

m_1	m_2	$J=5/2$, $M=+5/2$
+3/2	+1	1

m_1	m_2	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+3/2	0	2/5	3/5
+1/2	+1	3/5	−2/5

m_1	m_2	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+3/2	−1	1/10	2/5	1/2
+1/2	0	3/5	1/15	−1/3
−1/2	+1	3/10	−8/15	1/6

m_1	m_2	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1/2	−1	3/10	8/15	1/6
−1/2	0	3/5	−1/15	−1/3
−3/2	+1	1/10	−2/5	1/2

m_1	m_2	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
−1/2	−1	3/5	2/5
−3/2	0	2/5	−3/5

m_1	m_2	$J=5/2$, $M=-5/2$
−3/2	−1	1

$$Y_\ell^{-m}=(-1)^m Y_\ell^{m*}$$

$$d^{\,\ell}_{m,0}=\sqrt{\frac{4\pi}{2\ell+1}}\,Y_\ell^m\,e^{-im\phi}$$

$$\langle j_1 j_2 m_1 m_2 | j_1 j_2 J M\rangle$$
$$=(-1)^{J-j_1-j_2}\langle j_2 j_1 m_2 m_1 | j_2 j_1 J M\rangle$$

$$d^{\,j}_{m',m}=(-1)^{m-m'}d^{\,j}_{m,m'}=d^{\,j}_{-m,-m'}$$

$$d^{\,1}_{0,0}=\cos\theta \qquad d^{\,1/2}_{1/2,1/2}=\cos\frac{\theta}{2} \qquad d^{\,1/2}_{1/2,-1/2}=-\sin\frac{\theta}{2}$$

$$d^{\,1}_{1,1}=\frac{1+\cos\theta}{2} \qquad d^{\,1}_{1,0}=-\frac{\sin\theta}{\sqrt{2}} \qquad d^{\,1}_{1,-1}=\frac{1-\cos\theta}{2}$$

2×3/2

m_1	m_2	$J=7/2$, $M=+7/2$
+2	+3/2	1

m_1	m_2	$J=7/2$, $M=+5/2$	$J=5/2$, $M=+5/2$
+2	+1/2	3/7	4/7
+1	+3/2	4/7	−3/7

m_1	m_2	$J=7/2$, $M=+3/2$	$J=5/2$, $M=+3/2$	$J=3/2$, $M=+3/2$
+2	−1/2	1/7	16/35	2/5
+1	+1/2	4/7	1/35	−2/5
0	+3/2	2/7	−18/35	1/5

m_1	m_2	$J=7/2$, $M=+1/2$	$J=5/2$, $M=+1/2$	$J=3/2$, $M=+1/2$	$J=1/2$, $M=+1/2$
+2	−3/2	1/35	6/35	2/5	2/5
+1	−1/2	12/35	5/14	0	−3/10
0	+1/2	18/35	−3/35	−1/5	1/5
−1	+3/2	4/35	−27/70	2/5	−1/10

m_1	m_2	$J=7/2$, $M=-1/2$	$J=5/2$, $M=-1/2$	$J=3/2$, $M=-1/2$	$J=1/2$, $M=-1/2$
+1	−3/2	4/35	27/70	2/5	1/10
0	−1/2	18/35	3/35	−1/5	−1/5
−1	+1/2	12/35	−5/14	0	3/10
−2	+3/2	1/35	−6/35	2/5	−2/5

m_1	m_2	$J=7/2$, $M=-3/2$	$J=5/2$, $M=-3/2$	$J=3/2$, $M=-3/2$
0	−3/2	2/7	18/35	1/5
−1	−1/2	4/7	−1/35	−2/5
−2	+1/2	1/7	−16/35	2/5

m_1	m_2	$J=7/2$, $M=-5/2$	$J=5/2$, $M=-5/2$
−1	−3/2	4/7	3/7
−2	−1/2	3/7	−4/7

m_1	m_2	$J=7/2$, $M=-7/2$
−2	−3/2	1

3/2×3/2

m_1	m_2	$J=3$, $M=+3$
+3/2	+3/2	1

m_1	m_2	$J=3$, $M=+2$	$J=2$, $M=+2$
+3/2	+1/2	1/2	1/2
+1/2	+3/2	1/2	−1/2

m_1	m_2	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+3/2	−1/2	1/5	1/2	3/10
+1/2	+1/2	3/5	0	−2/5
−1/2	+3/2	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+3/2	−3/2	1/20	1/4	9/20	1/4
+1/2	−1/2	9/20	1/4	−1/20	−1/4
−1/2	+1/2	9/20	−1/4	−1/20	1/4
−3/2	+3/2	1/20	−1/4	9/20	−1/4

m_1	m_2	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1/2	−3/2	1/5	1/2	3/10
−1/2	−1/2	3/5	0	−2/5
−3/2	+1/2	1/5	−1/2	3/10

m_1	m_2	$J=3$, $M=-2$	$J=2$, $M=-2$
−1/2	−3/2	1/2	1/2
−3/2	−1/2	1/2	−1/2

m_1	m_2	$J=3$, $M=-3$
−3/2	−3/2	1

2×2

m_1	m_2	$J=4$, $M=+4$
+2	+2	1

m_1	m_2	$J=4$, $M=+3$	$J=3$, $M=+3$
+2	+1	1/2	1/2
+1	+2	1/2	−1/2

m_1	m_2	$J=4$, $M=+2$	$J=3$, $M=+2$	$J=2$, $M=+2$
+2	0	3/14	1/2	2/7
+1	+1	4/7	0	−3/7
0	+2	3/14	−1/2	2/7

m_1	m_2	$J=4$, $M=+1$	$J=3$, $M=+1$	$J=2$, $M=+1$	$J=1$, $M=+1$
+2	−1	1/14	3/10	3/7	1/5
+1	0	3/7	1/5	−1/14	−3/10
0	+1	3/7	−1/5	−1/14	3/10
−1	+2	1/14	−3/10	3/7	−1/5

m_1	m_2	$J=4$, $M=0$	$J=3$, $M=0$	$J=2$, $M=0$	$J=1$, $M=0$	$J=0$, $M=0$
+2	−2	1/70	1/10	2/7	2/5	1/5
+1	−1	8/35	2/5	1/14	−1/10	−1/5
0	0	18/35	0	−2/7	0	1/5
−1	+1	8/35	−2/5	1/14	1/10	−1/5
−2	+2	1/70	−1/10	2/7	−2/5	1/5

m_1	m_2	$J=4$, $M=-1$	$J=3$, $M=-1$	$J=2$, $M=-1$	$J=1$, $M=-1$
+1	−2	1/14	3/10	3/7	1/5
0	−1	3/7	1/5	−1/14	−3/10
−1	0	3/7	−1/5	−1/14	3/10
−2	+1	1/14	−3/10	3/7	−1/5

m_1	m_2	$J=4$, $M=-2$	$J=3$, $M=-2$	$J=2$, $M=-2$
0	−2	3/14	1/2	2/7
−1	−1	4/7	0	−3/7
−2	0	3/14	−1/2	2/7

m_1	m_2	$J=4$, $M=-3$	$J=3$, $M=-3$
−1	−2	1/2	1/2
−2	−1	1/2	−1/2

m_1	m_2	$J=4$, $M=-4$
−2	−2	1

$$d^{\,3/2}_{3/2,3/2}=\frac{1+\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,1/2}=-\sqrt{3}\frac{1+\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-1/2}=\sqrt{3}\frac{1-\cos\theta}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{3/2,-3/2}=-\frac{1-\cos\theta}{2}\sin\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,1/2}=\frac{3\cos\theta-1}{2}\cos\frac{\theta}{2}$$

$$d^{\,3/2}_{1/2,-1/2}=-\frac{3\cos\theta+1}{2}\sin\frac{\theta}{2}$$

$$d^{\,2}_{2,2}=\Big(\frac{1+\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{2,1}=-\frac{1+\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,0}=\frac{\sqrt{6}}{4}\sin^2\theta$$

$$d^{\,2}_{2,-1}=-\frac{1-\cos\theta}{2}\sin\theta$$

$$d^{\,2}_{2,-2}=\Big(\frac{1-\cos\theta}{2}\Big)^2$$

$$d^{\,2}_{1,1}=\frac{1+\cos\theta}{2}(2\cos\theta-1)$$

$$d^{\,2}_{1,0}=-\sqrt{\frac{3}{2}}\sin\theta\cos\theta$$

$$d^{\,2}_{1,-1}=\frac{1-\cos\theta}{2}(2\cos\theta+1)$$

$$d^{\,2}_{0,0}=\Big(\frac{3}{2}\cos^2\theta-\frac{1}{2}\Big)$$

Figure 43.1: The sign convention is that of Wigner (*Group Theory*, Academic Press, New York, 1959), also used by Condon and Shortley (*The Theory of Atomic Spectra*, Cambridge Univ. Press, New York, 1953), Rose (*Elementary Theory of Angular Momentum*, Wiley, New York, 1957), and Cohen (*Tables of the Clebsch-Gordan Coefficients*, North American Rockwell Science Center, Thousand Oaks, Calif., 1974).

Wigner-Eckart theorem for electric dipole transitions

Stating the theorem:

$$\langle \gamma' j' m' | \hat{T}^{(k)}_q | \gamma, j, m \rangle = \frac{\langle \gamma' j' \| \hat{T}^{(k)} \| \gamma, j \rangle}{\sqrt{2j'+1}} \underbrace{\langle j', m' | k, q; j, m \rangle}_{\text{C-G part det's the branching ratios}}$$

For our case $\hat{T}^{(k)}_q \xrightarrow[\substack{k=1 \\ q=0,\pm 1}]{\vec{r}\cdot\hat{\varepsilon}} \sum_{q=-1}^{1} r_{-q}\, \varepsilon_q$ → det's light pol.

For d states, $l=2 \Rightarrow j'=2$,

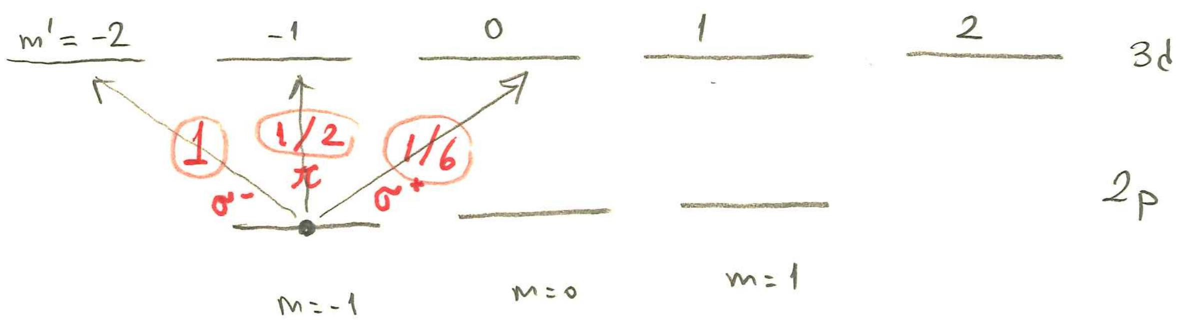
p " $l=1 \Rightarrow j=1$

Relative Branching Ratios : $\left| \langle j'=2, m' | 1, q; j=1, m=-1 \rangle \right|^2$

According to addition of angular momenta (C-G) only possibilities:

Use 1x1 sector in C-G table: $\langle 2,-2 | 1,-1; 1,-1 \rangle = 1$

$\langle 2,-1 | 1,0; 1,-1 \rangle = 1/\sqrt{2}$, $\langle 2,0 | 1,1; 1,-1 \rangle = 1/\sqrt{6}$



Thomas-Reiche-Kuhn Sum Rule

Without loss of generality, let the polarization of incident field be in the x-dir., i.e., $\hat{\epsilon} = \hat{x}$.

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_0 + \hat{\mathcal{H}}_{rad}(t)$$

$$= \frac{p^2}{2\mu} + V(r) - \frac{e\vec{A}\cdot\vec{p}}{\mu}$$

Within the standard perturbative framework, the atomic states are due to $\mathcal{H}_0$ part, $\mathcal{H}_0|i\rangle = \hbar\omega_i|i\rangle$, and the time-dependent $\mathcal{H}_{rad}(t)$ part leads to transitions among these $\{|i\rangle\}$ states.

Using $\mathcal{H}_0$ we shall show that

$$[x, \underbrace{[x,\mathcal{H}_0]}_{\frac{i\hbar}{\mu}p_x}] = -\frac{\hbar^2}{\mu},$$

Thus, $\langle i|[x,[x,\mathcal{H}_0]]|i\rangle = \langle i|x^2\mathcal{H}_0 + \mathcal{H}_0 x^2 - 2x\mathcal{H}_0 x|i\rangle$

E_i E_i

$$\langle i|-\frac{\hbar^2}{\mu}|i\rangle = 2\left[E_i\langle i|x.x|i\rangle - \langle i|x.\mathcal{H}_0.x|i\rangle\right]$$

$\sum_f |f\rangle\langle f|$ $\sum_f |f\rangle\langle f|$ $\sum_{f'} |f'\rangle\langle f'|$

and $\langle f|f'\rangle = \delta_{ff'}$

$$\Rightarrow -\frac{\hbar^2}{\mu} = 2\sum_f\left[\underbrace{\langle i|x|f\rangle E_i\langle f|x|i\rangle}_{E_i|\langle i|x|f\rangle|^2} - \underbrace{\langle i|x|f\rangle E_f\langle f|x|i\rangle}_{E_f|\langle i|x|f\rangle|^2}\right]$$

$$\Rightarrow \quad \frac{2\mu}{\hbar} \sum_f \omega_{fi} \left| \langle f | x | i \rangle \right|^2 = 1 \quad \ldots$$ Thomas - Reiche - Kuhn Sum Rule

As $\left| \langle f | \vec{r} | i \rangle \right|^2 = \left| \langle f | x | i \rangle \hat{x} + \langle f | y | i \rangle \hat{y} + \langle f | z | i \rangle \hat{z} \right|^2 = \left| \langle f | x | i \rangle \right|^2 + \left| \langle f | y | i \rangle \right|^2 + \left| \langle f | z | i \rangle \right|^2 \equiv \frac{1}{3} \left| \langle f | x | i \rangle \right|^2$, let

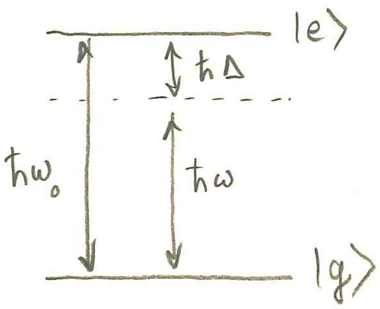
$$f_{fi} \equiv \omega_{fi} \frac{\frac{1}{3} \left| \langle f | \vec{r} | i \rangle \right|^2}{\frac{\hbar}{2\mu}} \quad \ldots$$ Oscillator Strength of transition $i \to f$

This is a measure of spatial overlap bet. $|i\rangle$ & $|f\rangle$

$\therefore$ we have $\sum_f f_{fi} = 1$ ... also known as f-sum rule (osc. str.)

Light Shift (AC Stark Effect)

Here, we consider the coupling bet. a 2-level atom $\{|g\rangle, |e\rangle\}$ with the photonic states $\{|n+1\rangle, |n\rangle\}$. Due to coupling bet. them a so-called **dressed-atom** emerges with different energies.



bare atom under red-detuned light

$$\Delta = \omega_0 - \omega$$

For this definition, $\Delta > 0 \Rightarrow$ red detuned radiation

Considering also the photon energies, under no coupling ($\mathcal{V} = 0$)

$$|e\rangle|n\rangle \longrightarrow E_e + n\hbar\omega$$

$$|g\rangle|n+1\rangle \longrightarrow E_g + (n+1)\hbar\omega$$

$$-$$

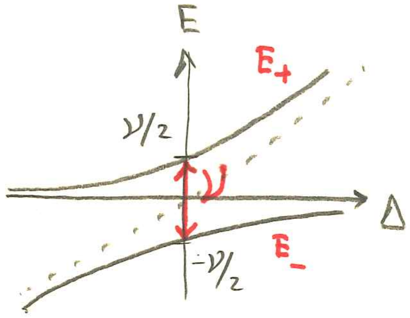
$$\underbrace{(E_e - E_g)}_{\hbar\omega_0} - \hbar\omega = \hbar\Delta$$

Under light-atom coupling the atomic subspace (i.e., suppressing photonic part) $(\hbar \to 1)$

$$\begin{bmatrix} \Delta & -\mathcal{V}/2 \\ -\mathcal{V}/2 & 0 \end{bmatrix} \begin{bmatrix} C_e \\ C_g \end{bmatrix}$$

a) Eigenvalues:
$$\begin{vmatrix} \Delta - E & -\nu/2 \\ -\nu/2 & -E \end{vmatrix} = 0$$

$$E^2 - \Delta E - \frac{\nu^2}{4} = 0 \ , \qquad E_\pm = \frac{\Delta \pm \sqrt{\Delta^2+\nu^2}}{2}$$



b) Eigenvectors: $|+\rangle$, $|-\rangle$

$$|+\rangle = C_{e+}|e\rangle + C_{g+}|g\rangle \qquad \begin{bmatrix} \Delta - E_+ & -\nu/2 \\ -\nu/2 & -E_+ \end{bmatrix}\begin{bmatrix} C_{e+} \\ C_{g+} \end{bmatrix} = 0$$

$$\Rightarrow \frac{\Delta - \sqrt{\Delta^2+\nu^2}}{2}\, C_{e+} = \frac{\nu}{2}\, C_{g+} \quad \& \quad |C_{e+}|^2 + |C_{g+}|^2 = 1 \ \text{(Normalization)}$$

Let $\tan\theta \equiv \dfrac{\nu}{\Delta}$, $\Rightarrow \left(1 - \underbrace{\sqrt{1+\tan^2\theta}}_{\sec\theta}\right) C_{e+} = \tan\theta\, C_{g+}$

$$\Rightarrow \frac{\cos\theta - 1}{\sin\theta}\, C_{e+} = C_{g+} \ , \quad \left.\begin{matrix}\text{combine with} \\ \text{normalization}\end{matrix}\right\} \ |C_{e+}| = \frac{1}{\sqrt{1+\left(\frac{\cos\theta-1}{\sin\theta}\right)^2}}$$

Taking C_{e+} as purely real,

$$C_{e+} = \frac{\sin\theta}{\sqrt{2-2\cos\theta}} = \frac{2\sin\frac{\theta}{2}\cdot\cos\frac{\theta}{2}}{2\cdot\sin\frac{\theta}{2}} = \cos\frac{\theta}{2}$$

$$\Rightarrow C_{g+} = -\cos\frac{\theta}{2}\cdot\frac{1-\cos\theta}{\sin\theta} = -\sin\frac{\theta}{2}$$

half ∠'s

$$|+\rangle = \cos\frac{\theta}{2}|e\rangle - \sin\frac{\theta}{2}|g\rangle$$

$$|-\rangle = C_{e-}\,|e\rangle + C_{g-}\,|g\rangle$$

$$\begin{bmatrix} \Delta - E_- & -\nu/2 \\ -\nu/2 & -E_- \end{bmatrix} \begin{bmatrix} C_{e-} \\ C_{g-} \end{bmatrix} = 0 \qquad \begin{cases} \dfrac{\Delta + \sqrt{\Delta^2 + \nu^2}}{2}\, C_{e-} = \dfrac{\nu}{2}\, C_{g-} \\ |C_{e-}|^2 + |C_{g-}|^2 = 1 \end{cases}$$

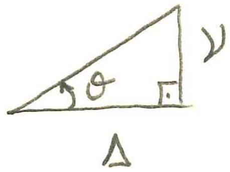
$$\left(1 + \sqrt{1 + \tan^2\theta}\right) C_{e-} = \tan\theta\, C_{g-} \qquad |C_{e-}| = \frac{1}{\sqrt{1 + \left(\frac{\cos\theta + 1}{\sin\theta}\right)^2}} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\cos\frac{\theta}{2}}$$

Taking C_{e-} as real

$$C_{e-} = \sin\frac{\theta}{2} \;; \qquad C_{g-} = \sin\frac{\theta}{2} \cdot \frac{1 + \cos\theta}{\sin\theta} = \sin\frac{\theta}{2} \cdot \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}} = \cos\frac{\theta}{2}$$

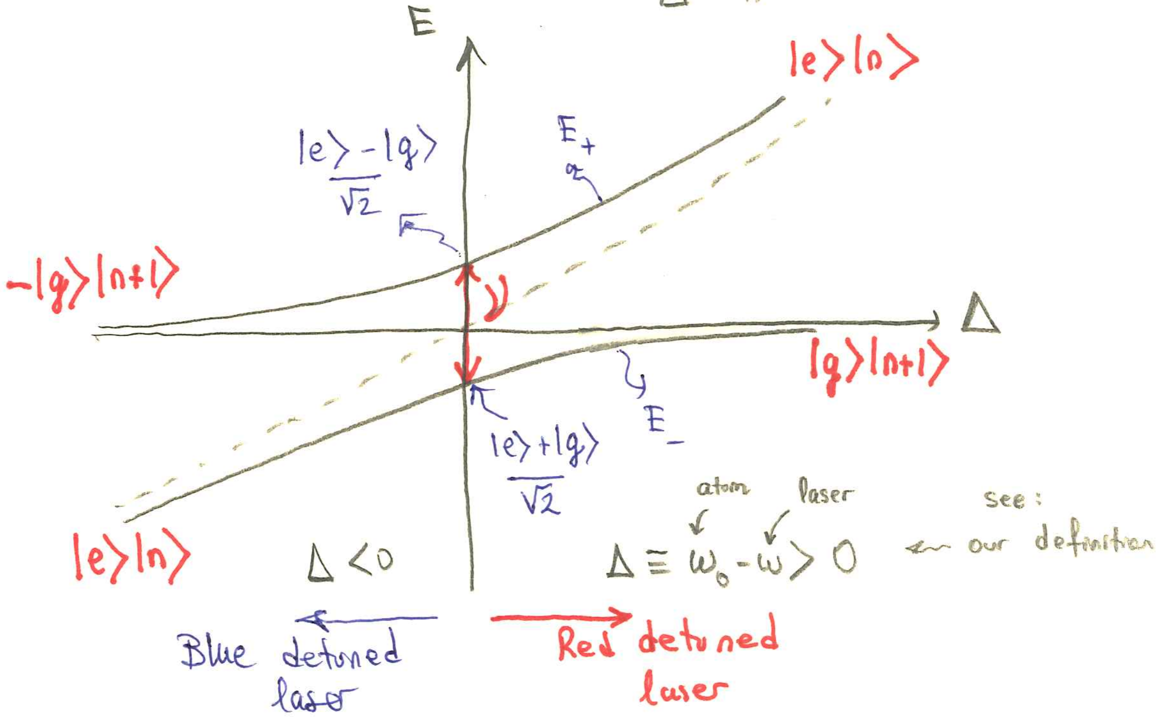
$$\Rightarrow \; |-\rangle = \sin\frac{\theta}{2}\,|e\rangle + \cos\frac{\theta}{2}\,|g\rangle$$; Check that $\langle +|-\rangle = 0$ ✓

Since $\tan\theta \equiv \dfrac{\nu}{\Delta}$

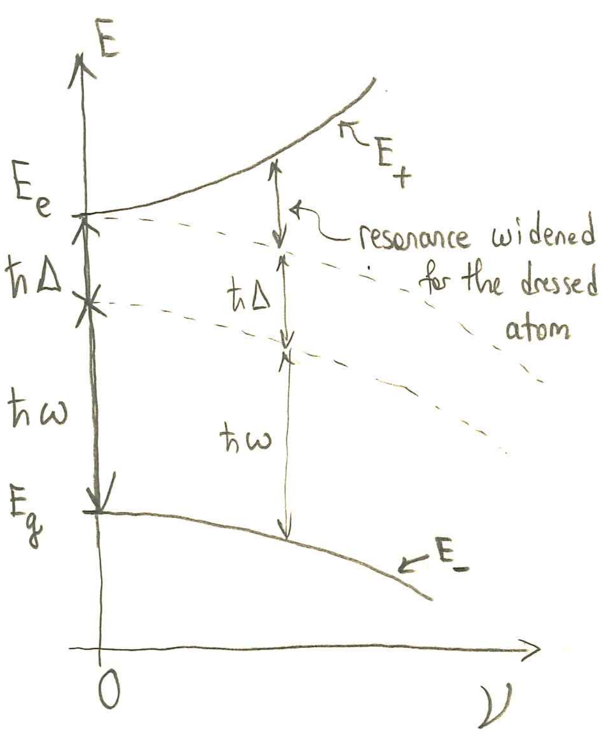


For a given ν value, as θ runs from $0 \rightarrow \frac{\pi}{2} \rightarrow \pi$

Δ 〃 〃 $+\infty \rightarrow 0 \rightarrow -\infty$

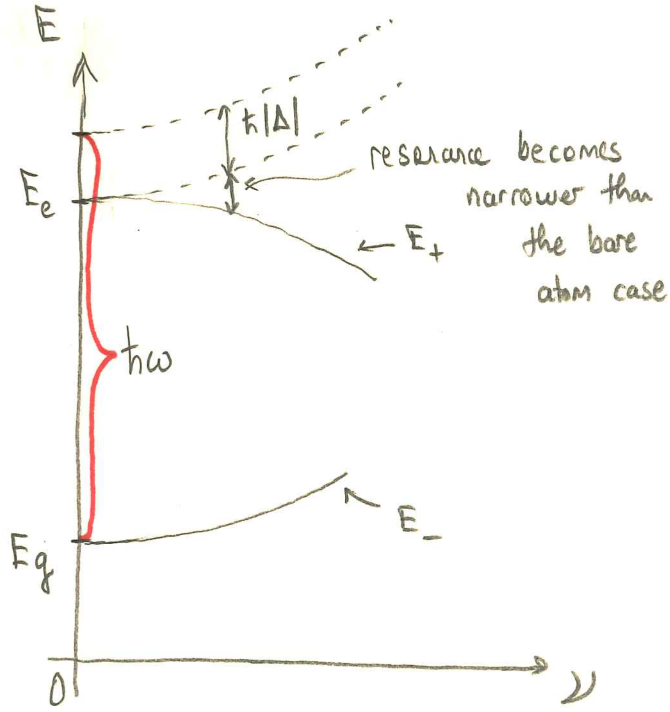


Physical Consequences of AC Stark Effect



red-detuned laser

$\Delta > 0$ (in our case)



blue-detuned laser

$\Delta < 0$

So, strong light coupling redresses the atom (a non-perturbative phenomenon) such that resonance becomes wider (narrower) than that wrt bare atom case under light-coupling ($\nu \neq 0$) for the case of red-detuned (blue-detuned) cases.